2018-2019 Yılı, MAT 311 Nümerik Analize Giriş Final Sınavı

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SORULAR

1. $f(x) = xe^{-x}$ olsun. $p_k = g(p_{k-1})$ Newton-Raphson formülü ile $p_0 = 0.2$ den başlayarak sırasıyla p_1, p_2, p_3 değerlerini bulunuz.

2. $$\begin{aligned} x_1 - x_2 - x_3 &= -1 \\ 2x_1 + x_2 - 2x_3 &= 1 \\ x_1 + x_2 + x_3 &= 3 \end{aligned}$$ denklem sistemini Gauss Eliminasyon yöntemi ile çözünüz.

3. e^{x^2} için $R_{2,2}(x)$ Pade yaklaşımını elde ediniz.

4. Taylor Serisi'ni kullanarak $f''(x_0) \approx \frac{f_1 - 2f_0 + f_{-1}}{h^2}$ merkezi fark formülünü elde ediniz. $(f_k = f(x_0 + kh))$

5. $P_1(x) = f_0 \frac{x - x_1}{x_0 - x_1} + f_1 \frac{x - x_0}{x_1 - x_0}$ Lagrange interpolasyon polinomunun $[x_0, x_1]$ aralığında integralini alarak yamuk kuralının formülünü kurunuz.

CEVAPLAR

1) $$P_k = g(P_{k-1}) = P_{k-1} - \frac{f(P_{k-1})}{f'(P_{k-1})} \quad ; \quad f'(x) = e^{-x} - xe^{-x}$$

$$= P_{k-1} - \frac{P_{k-1}e^{-P_{k-1}}}{e^{-P_{k-1}} - P_{k-1}e^{-P_{k-1}}} = \frac{\cancel{P_{k-1}e^{-P_{k-1}}} - P_{k-1}^2\cancel{e^{-P_{k-1}}} - \cancel{P_{k-1}e^{-P_{k-1}}}}{\cancel{e^{-P_{k-1}}}(1 - P_{k-1})}$$

$$= \frac{P_{k-1}^2}{P_{k-1} - 1}$$

$k=1$ için $\quad P_1 = \dfrac{P_0^2}{P_0 - 1} = \dfrac{0{,}2^2}{0{,}2 - 1} = -0{,}05$

$k=2$ için $\quad P_2 = \dfrac{P_1^2}{P_1 - 1} = \dfrac{(-0{,}05)^2}{-0{,}05 - 1} = -0{,}00238$

$k=3$ için $\quad P_3 = \dfrac{P_2^2}{P_2 - 1} = \dfrac{(-0{,}00238)^2}{-0{,}00238 - 1} = 0$

2) $$\left[\begin{array}{ccc|c} 1 & -1 & -1 & -1 \\ 2 & 1 & -2 & 1 \\ 1 & 1 & 1 & 3 \end{array}\right] \approx \left[\begin{array}{ccc|c} 1 & -1 & -1 & -1 \\ 0 & 3 & 0 & 3 \\ 0 & 2 & 2 & 4 \end{array}\right]$$

$$\approx \left[\begin{array}{ccc|c} 1 & -1 & -1 & -1 \\ 0 & 3 & 0 & 3 \\ 0 & 0 & 2 & 2 \end{array}\right] \begin{array}{l} \to x_1 - x_2 - x_3 = -1 \Rightarrow x_1 = -1+2 = 1 \\ \to 3x_2 = 3 \to x_2 = 1 \\ \to 2x_3 = 2 \to x_3 = 1 \end{array}$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

3) $$R_{2,2}(x) = \frac{b_0 + b_1 x + b_2 x^2}{1 + a_1 x + a_2 x^2}$$

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots \qquad e^{x^2} = 1 + x^2 + \frac{x^4}{2!} + \cdots$$

$$e^{x^2} - R_{22}(x) = O(x^5)$$

$$\left(1 + x^2 + \frac{x^4}{2!} + \cdots\right) - \frac{b_0 + b_1 x + b_2 x^2}{1 + a_1 x + a_2 x^2} = O(x^5)$$

$$(1 + a_1 x + a_2 x^2)\left(1 + x^2 + \frac{x^4}{2!} + \cdots\right) - b_0 - b_1 x - b_2 x^2 = O(x^5)$$

x^0: $1 - b_0 = 0 \to b_0 = 1$

x^1: $a_1 - b_1 = 0 \to b_1 = 0$

x^2: $1 + a_2 - b_2 = 0 \to b_2 = \frac{1}{2}$ $\qquad \Rightarrow R_{2,2}(x) = \dfrac{1 + \frac{1}{2}x^2}{1 - \frac{1}{2}x^2}$

x^3: $a_1 = 0$

x^4: $\frac{1}{2} + a_2 = 0 \to a_2 = -\frac{1}{2}$

4) $$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2} f''(x) + O(h^3)$$

$$f(x-h) = f(x) - h f'(x) + \frac{h^2}{2} f''(x) + O(h^3)$$

$$f(x+h) + f(x-h) = 2f(x) + h^2 f''(x) + O(h^3)$$

$$f''(x) = \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} + O(h)$$

$$f''(x) \approx \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}$$

$$f''(x_0) \approx \frac{f(x_0+h) - 2f(x_0) + f(x_0-h)}{h^2} = \frac{f_1 - 2f_0 - f_{-1}}{h^2}$$

5) $$\int_{x_0}^{x_1} P_1(x)\,dx = \int_{x_0}^{x_1} \left(f_0 \frac{x-x_1}{x_0-x_1} + f_1 \frac{x-x_0}{x_1-x_0} \right) dx$$

$$= \int_{x_0}^{x_1} f_0 \frac{x-x_1}{x_0-x_1}\,dx + \int_{x_0}^{x_1} f_1 \frac{x-x_0}{x_1-x_0}\,dx$$

$$= f_0 \int_{x_0}^{x_1} \frac{x-x_1}{x_0-x_1}\,dx + f_1 \int_{x_0}^{x_1} \frac{x-x_0}{x_1-x_0}\,dx$$

$$= \frac{f_0}{(x_0-x_1)} \left(\frac{x^2}{2} - x x_1 \right)_{x_0}^{x_1} + \frac{f_1}{(x_1-x_0)} \left(\frac{x^2}{2} - x x_0 \right)_{x_0}^{x_1}$$

$$= \frac{f_0}{x_0-x_1} \left[\frac{x_1^2}{2} - x_1^2 - \frac{x_0^2}{2} + x_0 x_1 \right] + \frac{f_1}{x_1-x_0} \left[\frac{x_1^2}{2} - x_1 x_0 - \frac{x_0^2}{2} + x_0^2 \right]$$

$$= \frac{f_0}{2(x_0-x_1)} \left(-x_1^2 - x_0^2 + 2x_0 x_1 \right) + \frac{f_1}{2(x_1-x_0)} \left(x_1^2 + x_0^2 - 2x_1 x_0 \right)$$

$x_1 = x_0 + h$

$$= \frac{f_0}{2(\cancel{x_0} - \cancel{x_0} - h)} \left[-\cancel{x_0^2} - 2\cancel{x_0}h - h^2 - \cancel{x_0^2} + 2x_0(\cancel{x_0} + \cancel{h}) \right]$$

$$+ \frac{f_1}{2(\cancel{x_0} + h - \cancel{x_0})} \left[\cancel{x_0^2} + 2\cancel{x_0}h + h^2 + \cancel{x_0^2} - 2(\cancel{x_0} + \cancel{h})x_0 \right]$$

$$= \frac{f_0}{-2h} \cdot -h^2 + \frac{f_1}{2h} h^2 = \frac{h}{2}(f_0 + f_1)$$